On a Problem of Rohrbach
For Finite Groups

Xingde Jia*
Department of Mathematics
Southwest Texas State University
San Marcos, Texas 78666. USA
e-mail: jia@swt.edu

July 1992

---

*This work was supported in part by an SWTSU Summer Research Grant

Abstract

Let $h \geq 2$ be any integer. In this paper, it is proved that, for any nonnegative real numbers $\alpha_1, \ldots, \alpha_h$ with $\alpha_1 + \cdots + \alpha_h = 1$, every finite solvable group G contains h subsets $A_1, \ldots, A_h$ such that

(a) $A_1 \cdots A_h = G$,
and

(b) $|A_i| \leq 2|G|^{\alpha_i}$ for $i = 1, \ldots, h$.

In particular, it is proved that every finite solvable group G has a "thin" basis A of order h such that $|A| \leq 2h|G|^{1/h}$. This answers an old question of Rohrbach in the solvable case. It is also proved in this paper that there exists a constant $c = c(h)$ such that *every* finite group has a basis A of order h such that $|A| \leq c|G|^{1/h}$ provided that the class of all finite simple groups is h-decomposable.

1 Introduction

Let G be a finite group. Let $h \geq 2$, and let $A_1, \ldots, A_h$ be subsets of G. Denote by $A_1 \cdots A_h$ the set of all elements of G that can be written in the form $a_1 \cdots a_h$, where $a_i \in A_i$ for $i = 1, \ldots, h$. When $A = A_1 = \cdots = A_h$, let $A^h = A_1 \cdots A_h$. A subset A of G is a *basis of order h for G* if $A^h = G$.

Let $|S|$ denote the cardinality of the set S. If A is a basis of order h for the finite group G, then

$$|A| \geq |G|^{1/h}.$$

In 1937, Rohrbach [9] asked if, for every $h \geq 2$, there exists a constant $c = c(h)$ such that every finite group G has a "thin" basis of order h such that

$$|A| \leq c|G|^{1/h}.$$

Rohrbach observed that such thin bases exist for finite cyclic groups, but the general case still remains unsolved. Cherly [2] proved that every finite abelian group G of order n has a basis A of order two such that

$$|A| \leq 2\sqrt{n \log n} + 2.$$

Recently, Jia [5,6] solved the Rohrbach's problem in the nilpotent case. He proved that every finite abelian group G has a basis A of order h such that

$$|A| \leq c_1|G|^{1/h},$$

where $c_1 = h(1 + 2^{-1/h})^{h-1}$, and that every finite nilpotent group G has a basis A of order h such that

$$|A| \leq c_2|G|^{1/h},$$

where $c_2 = h \cdot 2^{h-1}$.

In the general case, Bertram and Herzog [1] proved that for every $\delta > 0$ and for "almost all" n, every finite group G of order n has a basis A of order two such that[1]

$$|A| \leq |G|^{1/2+\delta}.$$

Recently, Nathanson [8] proved that every finite group G of order n has a basis A of order two such that

$$|A| \leq 2\sqrt{n \log n} + 2,$$

and, for every $h \geq 3$ and $\delta > 0$, there exists $M = M(h, \delta)$ such that every finite group G of order $n \geq M$ has a basis A of order h such that

$$|A| \leq (h + \delta)(n \log n)^{1/h}.$$

Very recently, Kozma and Lev [7] completely solved Rohrbach's problem in the case $h = 2$. They proved that every finite group G has a basis A of order two such that

$$|A| \leq \frac{4}{\sqrt{3}}\sqrt{|G|}.$$

In this paper, we shall show that, for every $h \geq 2$, every finite solvable group G has a basis A of order h such that

$$|A| \leq 2h|G|^{1/h}.$$

This solves Rohrbach's problem in the solvable case. The famous theorem of Feit and Thompson [3] asserts that every finite group of odd order is solvable. Therefore, as a consequence of the result in this paper, every finite group G of odd order has a basis A of order h such that

$$|A| \leq 2h|G|^{1/h}.$$

A class of finite groups is said to be be *h-decomposable* if there exists an absolute constant $c = c(h)$ such that, for any nonnegative real numbers $\alpha_1, \dots, \alpha_h$ with $\alpha_1 + \cdots + \alpha_h = 1$, every finite group in this class contains h subsets $A_1, A_2, \dots, A_h$ of G such that

(i) $A_1 A_2 \cdots A_h = G$, and

(i) $|A_i| = c_i n^{\alpha_i} (i = 1, 2, \dots, h)$ with $c_1 + c_2 + \cdots + c_h \leq c$.

In this paper, we shall prove that the class of all finite solvable groups is h-decomposable with $c = 2h$. In the last section of this paper, we shall prove that if the class of all finite simple groups is h-decomposable then the class of all finite group is h-decomposable.

---

[1] With a refinement of the result used in their proof, the Bertram-Herzog's method can imply the following slightly stronger result: Given any function f(n) which tends to infinity as slowly as we like, for almost all n, every finite group G of order n has a basis A of order two such that

$$|A| \leq \sqrt{n f(n) \log \log n}.$$

2 The Solvable Case

In this section, we shall prove the following result.

Theorem 1 *Let* $h \geq 2$ *be an integer. Let* $\alpha_1, \ldots, \alpha_h$ *be nonnegative real numbers with* $\alpha_1 + \cdots + \alpha_h = 1$. *Then every finite solvable group* G *of order* n *contains* h *subsets* $A_1, A_2, \ldots, A_h$ *of* G *such that*

(*i*) $A_1 A_2 \cdots A_h = G$, *and*

(*i*) $|A_i| \leq 2n^{\alpha_i}$ *for* $i = 1, 2, \ldots, h$.

Proof. Let G be any finite solvable group of order n. Then G possesses a composition series of normal subgroups

$$G = G_o \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_r \rhd G_{r+1} = E$$

such that every factor group G_i/G_{i+1} cyclic and of prime order p_i. Thus $n = p_0 \cdots p_r$.

For each $0 \leq i \leq r$, let $a_i G_{i+1}$ be a generator of the cyclic group G_i/G_{i+1} where $a_i \in G_i$. It is easy to show that every element $x \in G$ can be written in the form

$$x = a_0^{i_0} a_1^{i_1} \cdots a_r^{i_r},$$

where $0 \leq i_j \leq p_j - 1$ for $j = 0, 1, \ldots, r$. Noting that the order of G is $n = p_0 \cdots p_r$, we see that the above representation of x is unique. Define, for each $0 \leq i \leq r$,

$$U_i = \{e, a_i, a_i^2, \ldots, a_i^{p_i - 1}\},$$

where e is the identity element in G. Then

$$G = U_0 U_1 \cdots U_r.$$

Let $\alpha_1, \ldots, \alpha_h$ be any h nonnegative real numbers such that $\alpha_1 + \cdots + \alpha_h = 1$. We may assume without loss of generality that every $\alpha_i > 0$. We shall construct A_k's inductively. Suppose that

$$p_0 \cdots p_{\mu-1} < n^{\alpha_1} \leq p_0 \cdots p_\mu$$

for some integer $0 \leq \mu \leq r$. Let

$$m_1 = \left\lceil \frac{n^{\alpha_1}}{p_0 \cdots p_{\mu-1}} \right\rceil.$$

Define

$$A_1 = U_0 \cdots U_{\mu-1} R_1,$$

where

$$R_1 = \{e, a_\mu, a_\mu^2, \ldots, a_\mu^{m_1 - 1}\}.$$

Then

$$|A_1| \leq |U_0| \cdots |U_{\mu-1}||R_1| = p_0 \cdots p_{\mu-1} m_1 \leq 2n^{\alpha_1}.$$

Suppose that we have constructed $A_1, A_2, \ldots, A_k (1 \leq k \leq h-1)$ with the following properties:

(a) $|A_i| \leq 2n^{\alpha_i}$ for $i = 1, 2, \ldots, k$; and

(b) If $p_0 \cdots p_{\nu-1} < n^{\alpha_1 + \cdots + \alpha_k} \leq p_o \cdots p_\nu$, then

$$A_1 A_2 \cdots A_k \supseteq U_0 U_1 \cdots U_{\nu-1} R_k,$$

where

$$R_k = \{e, a_\nu, \ldots, a_\nu^{m_k - 1}\}, \quad \text{and} \quad m_k = \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_k}}{p_0 \cdots p_{\nu-1}} \right\rceil.$$

We now construct A_{k+1}. Suppose that

$$p_0 \cdots p_{\theta-1} < n^{\alpha_1 + \cdots + \alpha_{k+1}} \leq p_0 \cdots p_\theta.$$

It is clear that $\theta \geq \nu$. We break our construction into two cases.

CASE I. $\theta = \nu$. Let $t = \lceil n^{\alpha_{k+1}} \rceil$, and define

$$A_{k+1} = \{e, a_\theta^{m_k}, a_\theta^{2m_k}, \ldots, a_\theta^{(t-1)m_k}\}.$$

Then

$$|A_{k+1}| = t = \lceil n^{\alpha_{k+1}} \rceil \leq 2n^{\alpha_{k+1}}.$$

Let

$$m_{k+1} = \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_{k+1}}}{p_0 \cdots p_{\theta-1}} \right\rceil,$$

and define

$$R_{k+1} = \{e, a_\theta, \ldots, a_\theta^{m_{k+1}-1}\}.$$

We need to show that

$$A_1 \cdots A_{k+1} \supseteq U_0 \cdots U_{\theta-1} R_{k+1}. \tag{1}$$

It follows the hypothesis (b) that we only need to show that

$$R_k A_{k+1} \supseteq R_{k+1}.$$

Let u be any integer such that $0 \leq u \leq t m_k - 1$, then

$$u = v m_k + w, \quad \text{where} \quad 0 \leq v \leq t-1, \ 0 \leq w \leq m_k - 1.$$

Therefore,

$$a_\theta^u = a_\theta^w a_\theta^{v m_k} \in R_k A_{k+1},$$

which implies that

$$R_k A_{k+1} \supseteq \{e, a_\theta, \ldots, a_\theta^{tm_k - 1}\}.$$

Hence (1) follows from the fact $\theta = \nu$ and

$$tm_k = \lceil n^{\alpha_{k+1}} \rceil \cdot \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_k}}{p_0 \cdots p_{\nu-1}} \right\rceil \geq \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_{k+1}}}{p_0 \cdots p_{\theta-1}} \right\rceil = m_{k+1}$$

CASE II. $\theta > \nu$. Let $t = \lfloor p_\nu / m_k \rfloor$. Define

$$L_{k+1} = \{e, a_\theta^{m_k}, \ldots, a_\nu^{tm_k}\}.$$

Then a similar argument as we used above shows that

$$R_k L_{k+1} = \{e, a_\nu, \ldots, a_\nu^{(t+1)m_k - 1}\}.$$

Since

$$(t+1)m_k - 1 \geq \frac{p_\nu}{m_k} \cdot m_k - 1 = p_\nu - 1,$$

we see that

$$R_k L_{k+1} = U_\nu. \tag{2}$$

Let m_{k+1} and R_{k+1} be as we defined in the Case I. Define

$$A_{k+1} = L_{k+1} U_{\nu+1} \cdots U_{\theta-1} R_{k+1}.$$

It follows from (2) that

$$A_1 \cdots A_{k+1} \supseteq U_0 \cdots U_{\theta-1} R_{k+1}.$$

Noting that

$$m_k = \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_k}}{p_0 \cdots p_{\nu-1}} \right\rceil \geq \frac{n^{\alpha_1 + \cdots + \alpha_k}}{p_0 \cdots p_{\nu-1}},$$

we have

$$\begin{aligned}
|A_{k+1}| &\leq |L_{k+1}||U_{\nu+1}| \cdots |U_{\theta-1}||R_{k+1}| \\
&= tp_{\nu+1} \cdots p_{\theta-1} m_{k+1} \\
&= \left\lfloor \frac{p_\nu}{m_k} \right\rfloor \cdot p_{\nu+1} \cdots p_{\theta-1} \cdot \left\lceil \frac{n^{\alpha_1 + \cdots + \alpha_k + 1}}{p_0 \cdots p_{\nu-1}} \right\rceil \\
&\leq \frac{2}{m_k} \cdot \frac{n^{\alpha_1 + \cdots + \alpha_k + 1}}{p_0 \cdots p_{\nu-1}} \\
&\leq 2 \cdot \frac{p_0 \cdots p_{\nu-1}}{n^{\alpha_1 + \cdots + \alpha_k}} \cdot \frac{n^{\alpha_1 + \cdots + \alpha_k + 1}}{p_0 \cdots p_{\nu-1}} \\
&= 2n^{\alpha_{k+1}}.
\end{aligned}$$

Thus, we can construct $A_1, \ldots, A_h$ such that

$$|A_i| \leq 2n^{\alpha_i} \ for \ i = 1, 2, \ldots, h.$$

Since

$$p_0 \cdots p_{r-1} < n^{\alpha_1+\cdots+\alpha_h} = n = p_0 \cdots p_r,$$

we see that

$$m_h = \left\lceil \frac{n^{\alpha_1+\cdots+\alpha_h}}{p_0 \cdots p_{r-1}} \right\rceil = p_r,$$

which implies that

$$R_h = \{e, a_r, \ldots, a_r^{p_r-1}\} = U_r.$$

Therefore,

$$A_1 \cdots A_h \supseteq U_0 \cdots U_{r-1} R_h = U_0 \cdots U_{r-1} U_r = G.$$

The proof is complete.

As a consequence of this theorem, we have

Theorem 2 *For every $h \geq 2$, every finite solvable group G has a basis A of order h such that $|A| \leq 2h|G|^{1/h}$.*

Proof. Let G be any finite solvable group. Applying Theorem 1 to G with $\alpha_i = 1/h$ for $i = 1, \ldots, h$, we have subsets $A_1, \cdots, A_h$ of G such that $A_1 \cdots A_h = G$ and $|A_i| \leq 2|G|^{1/h}$ for $i = 1, \ldots, h$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_h$. Then A is a basis of order h for G and $|A| \leq 2h|G|^{1/h}$. The proof is complete.

Feit and Thompson [3] proved in 1963 that every finite group of odd order is solvable. The following theorems follows immediately from Theorems 1 and 2 and Feit-Thompson Theorem.

Theorem 3 *Let $h \geq 2$ be an integer. Let $\alpha_1, \ldots, \alpha_h$ be nonnegative real numbers with $\alpha_1 + \cdots + \alpha_h = 1$. Then every finite group G of odd order n contains h subsets $A_1, A_2, \ldots, A_h$ of G such that*

1. *$A_1 A_2 \cdots A_h = G$, and*
2. *$|A_i| \leq 2n^{\alpha_i}$ for $i = 1, 2, \ldots, h$.*

Theorem 4 *For every $h \geq 2$, every finite group G of odd order has a basis A of order h such that $|A| \leq 2h|G|^{1/h}$.*

3 The General Case

Theorem 5 *Let $h \geq 2$ be any integer. If the class of all finite simple groups is h-decomposable, so is the class of all finite groups.*

Proof. Suppose the class of all finite simple groups is h-decomposable with the constant $c > 0$. Let G be any finite group of order n. Let

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_r \rhd G_{r+1} = E$$

be a composition series of normal subgroups, where the factor group G_i/G_{i+1} is a simple group of order n_i for $i = 0, 1, \ldots, r$. Then $n = n_0 n_1 \cdots n_r$.

Let $\alpha_1, \ldots, \alpha_h$ be any nonnegative real numbers such that $\alpha_1 + \cdots + \alpha_h = 1$. We shall show that there exist h subsets $A_1, \ldots, A_h$ of G such that *(i)* $A_i \cdots A_h = G$

(i) $|A_i| = c^2 |G|^{\alpha_i}$ for $i = 1, \ldots, h$

Let $N_i = n_0 \cdots n_i$ for $i = 0, \ldots, r$. Suppose that

$$N_{\mu_{k-1}} < n^{\alpha_1 + \cdots + \alpha_k} \leq N_{\mu_k} \tag{3}$$

for $k = 1, \ldots, h$. It is clear that

$$0 \leq \mu_1 \leq \cdots \leq \mu_h = r.$$

Let $k_0 = 0$ and $k_s = h$, and assume that

$$\begin{aligned} \mu_1 = \cdots = \mu_{k_1} < \cdots < \mu_{k_{i-1}+1} = \cdots = \mu_{k_i} & \\ < \cdots < \mu_{k_{s-1}+1} = \cdots = \mu_{k_s} = \mu_h. \end{aligned}$$

For each $i : 1 \leq i \leq s$, let

$$\begin{aligned} t_i &= k_i - k_{i-1}, \\ \nu_i &= \mu_{k_{i-1}+1}, \\ e_i &= \alpha_1 + \cdots + \alpha_{k_{i-1}+1}, \\ f_i &= \alpha_1 + \cdots + \alpha_{k_i}. \end{aligned}$$

Let i be any integer with $1 \leq i \leq s$. Define real numbers $\beta_j^{(i)}$ for $j = 1, \ldots, t_i + 1$ as follows:

$$\begin{aligned} n_{\nu_i}^{\beta_1^{(i)}} &= \frac{n^{e_i}}{N_{\nu_i - 1}}, \\ n_{\nu_i}^{\beta_j^{(i)}} &= n^{\alpha_{k_{i-1}+j}} \; for \; j = 2, \ldots, t_i, \\ n_{\nu_i}^{\beta_{t_i+1}^{(i)}} &= \frac{N_{\nu_i}}{n^{f_i}}. \end{aligned}$$

Note that $\beta_{t_s+1}^{(s)} = 0$. It follows from (3) that

$$1 < \frac{n^{\epsilon_i}}{N_{\nu_i - 1}} \leq n_{\nu_i},$$

which implies that every $\beta_1^{(i)} > 0$. For each $2 \leq j \leq t_i$, $\alpha_{k_{i-1}+j} \geq 0$ implies $\beta_j^{(i)} \geq 0$. From (3) and the fact that $\mu_{k_{i-1}+1} = \mu_{k_i}$, we see that

$$n_{\nu_i} > \frac{N_{\nu_i}}{n^{f_i}} \geq 1,$$

which implies that $\beta^{(s)}_{t_i+1} > 0$. Since

$$\begin{aligned}
n_{\nu_i}^{\beta_1^{(i)}+\cdots+\beta_{t_i+1}^{(i)}} &= \prod_{j=1}^{t_i+1} n_{\nu_i}^{\beta_j^{(i)}} \\
&= \frac{n^{\epsilon_i}}{N_{\nu_i-1}} \cdot \prod_{j=2}^{t_i} n^{\alpha_{k_{i-1}+j}} \cdot \frac{N_{\nu_i}}{n^{f_i}} \\
&= \frac{n^{\epsilon_i+\alpha_{k_{i-1}+2}+\cdots+\alpha_{k_i}}}{n^{f_i}} \cdot \frac{N_{\nu_i}}{N_{\nu_i-1}} \\
&= n_{\nu_i},
\end{aligned}$$

we have

$$\beta_1^{(i)} + \cdots + \beta_{t_i+1}^{(i)} = 1.$$

It is clear that $t_i + 1 \leq h$. Since G_{ν_i}/G_{ν_i+1} is a simple group order n_{ν_i}, and hence (t_i+1)-decomposable, G_{ν_i}/G_{ν_i+1} contains $t_i + 1$ subsets $B_1^{(i)}, \ldots, B_{t_i+1}^{(i)}$ such that

(i) $B_1^{(i)} \cdots B_{t_i+1}^{(i)} = G_{\nu_i}/G_{\nu_i+1}$;

(ii) $c_1^{(i)} + \cdots + c_{t_i+1}^{(i)} \leq c$ where $|B_j^{(i)}| = c_j^{(i)} n_{\nu_i}^{\beta_j^{(i)}}$ for $j = 1, \ldots, t_i + 1$.

Since $\beta^{(s)}_{t_s+1} = 0$, we may choose that

$$B^{(s)}_{t_s+1} = \{e\}. \tag{4}$$

Let $C_j^{(i)} \subseteq G_{\nu_i}$ be the set of representatives of $B_j^{(i)} \subseteq G_{\nu_i+1}$. Then

$$|C_j^{(i)}| = |B_j^{(i)}| \leq c n_{\nu_i}^{\beta_j^{(i)}} \tag{5}$$

for $j = 1, \ldots, t_i + 1$. Let

$$U_{\nu_i} = \prod_{j=1}^{t_i+1} C_j^{(i)}, \tag{6}$$

it then follows form (i) above that

$$U_{\nu_i} G_{\nu_i+1} = G_{\nu_i}.$$

For $\nu \neq nu_i$, let U_ν be a subset of G_ν such that $|U_\nu| = n_\nu$ and $U_\nu G_{\nu+1} = G_\nu$. Then it easy to show that

$$G = U_0 U_1 \cdots U_r. \tag{7}$$

We now define $A_1, \ldots, A_h$ as follows. For $1 \leq i \leq s$, let

$$\begin{aligned}
A_{k_{i-1}+1} &= C^{(i-1)}_{t_{i-1}+1} U_{\nu_{i-1}+1} \cdots U_{v_i-1} C_1^{(i)}, \\
A_{k_{i-1}+j} &= C_j^{(i)} \; for \; j = 2, \ldots, t_i,
\end{aligned}$$

where we assumed that $C^{(0)}_{t_0+1} = \{e\}$, and $\nu_0 = -1$. Then it follows from (5) and the definitions that

$$\begin{aligned}
|A_{k_{i-1}+1}| &\leq |C^{(i-1)}_{t_{i-1}+1}||U_{v_{i-1}+1}| \cdots |U_{v_i-1}||C^{(i)}_1| \\
&\leq cn_{\nu_{i-1}}^{\beta^{(i-1)}_{t_{i-1}+1}} \cdot n_{\nu_{i-1}+1} \cdots n_{\nu_{i-1}} \cdot cn_{\nu_i}^{\beta^{(i)}_1} \\
&= c^2 \frac{N_{\nu_{i-1}}}{n^{f_i-1}} \cdot n_{\nu_{i-1}+1} \cdots n_{\nu_i-1} \cdot \frac{n^{e_i}}{N_{\nu_i-1}} \\
&= c^2 n^{\alpha_{k_{i-1}+1}}; \\
|A_{k_{i-1}+j}| &= |C^{(i)}_j| \leq cn_{\nu_i}^{\beta^{(i)}_j} \\
&\leq cn^{\alpha_{k_{i-1}+j}} \ for\ j = 2, \ldots, t_i.
\end{aligned}$$

We may assume without loss of generality that $c \geq 1$, then $|A_m| \leq c^2 n^{\alpha_m}$ for $m = 1, 2, \ldots, h$. Therefore, we only need to show that $A_1 \cdots A_h = G$. It follows from (4), (6), and (7) that

$$\begin{aligned}
A_1 \cdots A_h &= \prod_{i=1}^{s} \prod_{j=1}^{t_i} A_{k_{i-1}+j} \\
&= \prod_{i=1}^{s} \left\{ C^{(i-1)}_{t_{i-1}+1} U_{\nu_{i-1}+1} \cdots U_{\nu_{i-1}} C^{(i)}_1 \cdot \prod_{j=2}^{t_i} C^{(i)}_j \right\} \\
&= \prod_{i=1}^{s} \left\{ C^{(i-1)}_{t_{i-1}+1} U_{\nu_{i-1}+1} \cdots U_{\nu_{i-1}} C^{(i)}_1 \cdot \prod_{j=2}^{t_i} C^{(i)}_j \right\} \cdot C^{(s)}_{t_s+1} \\
&= \prod_{i=1}^{s} \left\{ U_{\nu_{i-1}+1} \cdots U_{\nu_{i-1}} \cdot \prod_{j=1}^{t_i+1} C^{(i)}_j \right\} \\
&= \prod_{i=1}^{s} \{ U_{\nu_{i-1}+1} \cdots U_{\nu_i} \} \\
&= \prod_{\nu=1}^{r} U_\nu \\
&= G.
\end{aligned}$$

The proof is complete.

Corollary 1 *If the class of all finite simple groups is h-decomposable, then every finite group G has a basis A of order h such that*

$$|A| \leq c|G|^{1/h},$$

where $c = c(h)$ is a constant depending only on h.

Proof. This follows immediately from the above theorem.

References

1. E. A. Bertram and M. Herzog, On medium-size subgroups and bases of finite groups, *J. Combin. Theory Ser. A*, **57**(1991), 1-14.
2. J. Cherly, On complementary sets of group elements, *Arch. Math.*, **35**(1980), 313-318.
3. W. Feit and G. Thompson, Solvability of groups of odd order, *Pacific J. Math.*, **13**(1963), 775-1029.
4. D. Gorenstein, *Finite Groups*, Harper & Row, Publishers, New York, 1968.
5. X.-D. Jia, Thin bases for finite abelian groups, *J. Number Theory*, **36**(1990), 254-256.
6. X.-D. Jia, Thin bases for finite nilpotent groups, *J. Number Theory*, **41**(1992), 303-313.
7. G. Kozma and A. Lev., Bases and decomposition numbers of finite groups, *Arch. Math.*, **58**(1992), 417-424.
8. M.B. Nathanson, On a problem of Rohrbach for finite groups, *J. Number Theory*, **41**(1992), 69-76.
9. H. Rohrbach, Ein Beitrag zur additiven Zahlentheorie, *Math. Zeit.*, **42**(1937), 1-30.
10. H. Rohrbach, Anwendung eines Satzes der additiven Zahlentheorie auf eine grupentheoretische Grage, *Math. Zeit.*, **42**(1937), 538-542.