

# On the Exact Order of Asymptotic Bases and Bases for Finite Cyclic Groups

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*Dedicated to Professor Melvyn B. Nathanson*

**Abstract** Let  $h$  be a positive integer, and  $A$  a set of nonnegative integers.  $A$  is called an *exact asymptotic basis* of order  $h$  if every sufficiently large positive integer can be written as a sum of  $h$  not necessarily distinct elements from  $A$ . The smallest such  $h$  is called the *exact order* of  $A$ , denoted by  $g(A)$ . A subset  $A - F$  of an asymptotic basis of order  $h$  may not be an asymptotic basis of any order. When  $A - F$  is again an asymptotic basis, the exact order  $g(A - F)$  may increase. Nathanson [48] studied how much larger the exact order  $g(A - F)$  when finitely many elements are removed from an asymptotic basis of order  $h$ . Nathanson defines, for any given positive integers  $h$  and  $k$ ,

$$G_k(h) = \max_{\substack{A \\ g(A) \leq h}} \max_{F \in I_k(A)} g(A - F),$$

where  $I_k(A) = \{F \mid |F| = k \text{ and } g(A - F) < \infty\}$ . Many results have been proved since Nathanson's question was first asked in 1984. This function  $G_k(h)$  is also closely related to interconnection network designs in network theory. This paper is a brief survey on the this and few other related problems. G. Grekos [10] has a recent survey on a related problem.

## 1 Exact Asymptotic Bases

Let  $\mathbb{N}$  be the set of all nonnegative integers. Let  $h$  be a positive integer and  $A \subseteq \mathbb{N}$  a set of nonnegative integers. Let  $hA$  denote the set of all sums of  $h$  not necessarily distinct elements from  $A$ . For notations and concepts without definitions here, the reader is referred to the books by Nathanson [50, 51].

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**Definition 1.** A set  $A \subseteq \mathbb{N}$  of nonnegative integers is called an *exact asymptotic basis* of order  $h$  if  $\mathbb{N} - hA$  is a finite set. In other words,  $A$  is called an exact asymptotic basis of order  $h$  if every sufficiently large positive integer can be written as a sum of  $h$  not necessarily distinct elements from  $A$ . The smallest such  $h$  is called the *exact order* of  $A$ , and is denoted by  $g(A)$ .

Let  $h$  be any positive integer. Then  $A = \{1\} \cup \{nh \mid n \in \mathbb{N}\}$  is an exact asymptotic basis of order  $h$ . Let  $B$  be the set of all odd positive integers. Then  $B$  is not an exact asymptotic basis of any order  $h$  because the sum of  $h$  odd integers has the same parity as  $h$ . However,  $B$  with any additional even integer becomes an exact asymptotic basis of order 2.

As an example, we show that  $A = \bigcup_{k=0}^{\infty} (2^{2k}, 2^{2k+1}]$  is an exact asymptotic basis of order 3, where we use  $(a, b]$  to denote the set of integers  $x$  with  $a < x \leq b$ . First, for a large  $k$ ,  $n = 2^{2k} + 1$  cannot be written as a sum of two elements from  $A$ . Otherwise, say  $n = a + b$ , then  $a, b \leq 2^{2(k-1)+1}$ . Hence  $a + b \leq 2^{2k-1} + 2^{2k-1} \leq 2^{2k} < n$ , a contradiction. We then show that every large integer  $n$  not in  $A$  can be written as a sum of two elements from  $A$ . First assume  $n = 2^{2k+1} + 1$ . If  $k$  is even,  $2^k + 1 \in A$  and  $2^{k+1} \in A$ . Hence  $n = (2^k + 1) + 2^{k+1} \in 2A$ . If  $k$  is odd, then  $n = 2^k + (2^{k+1} + 1) \in 2A$ . Now assume  $2^{2k+1} + 2 \leq n \leq 2^{2k+2}$  for some  $k$ . Then we can rewrite  $n = (2^{2k} + s) + (2^{2k} + t)$  with  $1 \leq s < 2^{2k}$  and  $1 \leq t \leq 2^{2k}$ . Hence  $2^{2k} + s, 2^{2k} + t \in (2^{2k}, 2^{2k+1}] \subseteq A$ , which implies that  $n \in 2A$ . Now pick a large positive integer  $n$  and write  $n = m + 5$ . If  $m \notin A$ , then, as shown earlier,  $m$  is a sum of two elements in  $A$ . Hence  $n$  is a sum of three elements in  $A$ . If  $m = 2^{2k} + 1$ , then  $n = 6 + 2^{2k-1} + 2^{2k-1} \in 3A$ . If  $m \in [2^{2k} + 2, 2^{2k+1}] \subseteq A$ , then  $n = 5 + (2^{2k-1} + s) + (2^{2k-1} + t) \in 3A$ . Therefore, we proved that  $A$  is an exact asymptotic basis of order 3.

## 2 Subsets of Exact Asymptotic Bases

A subset of an exact asymptotic basis of order  $h$  may not be again an exact asymptotic basis of any order. For instance,  $A = \{0\} \cup \{1, 3, 5, \dots\}$  is an exact asymptotic basis of order 2. However,  $A - \{0\}$ , the set of all odd positive integers is not an exact asymptotic basis of any order. When a subset of an exact asymptotic basis of order  $h$  is an exact asymptotic basis, its exact order can be larger. Nathanson defined the following function to study how large the exact order of subsets of an exact asymptotic basis of order  $h$  when finite.

Let  $A$  be an asymptotic basis of order  $h$ . Given a positive integer  $k$ , let

$$I_k(A) = \{F \subseteq A \mid |F| = k, \text{ and } A - F \text{ is an exact asymptotic basis}\}.$$

Define

$$G_k(h) = \max_{\substack{A \\ g(A) \leq h}} \max_{F \in I_k(A)} g(A - F),$$

where the first maximum is taken over all exact asymptotic bases of order at most  $h$ . Nathanson [48] proved in 1984 the following theorem.

**Theorem 1 (Nathanson [48], 1984).** *For  $h > k$ ,*

$$G_k(h) \geq \left( \left\lfloor \frac{h}{k+1} \right\rfloor + 1 \right)^{k+1} - 1 \approx \left( \frac{h}{k+1} \right)^{k+1}.$$

In 1988, Jia [22] improved Nathanson's lower bound to:

$$G_k(h) \geq \frac{4}{3} \left( \frac{h}{k+1} \right)^{k+1} + O(h^k) \quad \text{as } h \rightarrow \infty.$$

Later this has been further improved [24] to

$$G_k(h) \geq (k+1) \left( \frac{k+1}{k+2} \right)^k \left( \frac{h}{k+1} \right)^{k+1} + O(h^k) \quad \text{as } h \rightarrow \infty.$$

Let  $A$  be a set of nonnegative integers. The *lower density* of  $A$  is defined by

$$d(A) = \liminf_{m \rightarrow \infty} \frac{A(m)}{m},$$

where  $A(m) = |\{a \in A \mid 0 < a \leq m\}|$ . The following version of Kneser's Theorem [37] is useful in establishing upper bounds for  $G_k(h)$ .

**Theorem 2 (Kneser [37], 1953).** *Let  $C = A_1 + \cdots + A_n$ . Then either*

$$d(C) \geq d(A_1) + \cdots + d(A_n)$$

*or  $C$  is equal to, with at most finitely many exceptions, a residue class modulo  $g$  for some positive integer  $g$ .*

By using Kneser's Theorem for the upper bounds, Nash [46] proved in 1985 that

$$G_k(2) = 2k + 2 \quad \text{for all } k \geq 1.$$

In the general case, as  $k \rightarrow \infty$  for any given  $h \geq 1$ , we proved [24] that

$$\begin{aligned} G_k(h) + 1 &\geq 2 \left( \frac{k}{h-1} \right)^{h-1} + (4h-5) \left( \frac{k}{h-1} \right)^{h-2} + O(k^{h-3}), \\ G_k(h) + 1 &\leq \frac{2}{(h-1)!} k^{h-1} + \frac{h-1}{(h-2)!} k^{h-2} + O(k^{h-3}). \end{aligned}$$

When  $h = 2$ , this is Nash's formula for  $G_k(2)$ . Farhi [7] studied the exact order  $g(A - F)$  in terms of some parameters of the set  $F$  other than simply the cardinality of  $F$ . Define

$$d = \frac{\text{diam}(F)}{\text{gcd}\{x - y \mid x, y \in A\}}.$$

Farhi [7] proved in 2008 that

$$g(A - F) \leq \frac{h(h+3)}{2} + \frac{dh(h-1)(h+4)}{6}$$

for any exact asymptotic basis  $A$  of order  $h$ . In particular, if  $F$  is an arithmetic progression, then

$$g(A - F) \leq \frac{h(h+3)}{2} + \frac{(|F| - 1)h(h-1)(h+4)}{6}.$$

These lower bounds are better in many cases.

### 3 Exact Order of Asymptotic Bases

A set  $A$  of nonnegative integers is called an *asymptotic basis* of order  $h$  if every sufficiently large integer can be written as a sum of at most  $h$  not necessarily distinct elements from  $A$ . If we use notation

$$h^0A = \bigcup_{s=1}^h sA,$$

then  $A$  is an asymptotic basis of order  $h$  if and only if  $h^0A$  contains all sufficient large integers.

The set of all positive odd integers is an asymptotic basis of order 2 while it is not an exact asymptotic basis of any order. The following theorem of Erdős and Graham provides a necessary and sufficient condition for an asymptotic basis to be an exact asymptotic basis.

**Theorem 3 (Erdős and Graham [5], 1980).** *Assume that  $A$  is an asymptotic basis. Then  $A$  is an exact asymptotic basis if and only if*

$$\gcd\{a - a' : a, a' \in A\} = 1.$$

A more general theorem is proved by Nash and Nathanson [47] in 1985.

**Theorem 4 (Nash and Nathanson [47], 1985).** *If  $A$  is an asymptotic basis containing at most a finite number of negative terms such that*

$$\gcd\{a - a' : a, a' \in A\} = d,$$

*then there exists a positive integer  $q$  such that every term of an infinite arithmetic progression with difference  $d$  can be written as a sum of exactly  $q$  elements in  $A$ .*

Let  $A = \bigcup_{k=0}^{\infty} [2^{2k}, 2^{2k+1}]$ . Since every large integer  $n$  not in  $A$  can be written as a sum of two elements from  $A$ ,  $A$  is an asymptotic basis of order two. Note that  $A$

is an exact asymptotic basis of order 3. A natural question is how much bigger the exact order is when an asymptotic basis of order  $h$  is also an exact asymptotic basis.

**Theorem 5 (Nathanson [48], 1984).** *Let  $h \geq 2$  be an integer. Then*

$$G_1(h) = \max_A \{g(A) \mid A \text{ is an asymptotic basis of order } h\} \quad (1)$$

*Proof.* Denote the right hand side of (1) by  $t(h)$ . We only need to prove  $G_1(h) = t(h)$ .

Assume that  $A$  is an exact asymptotic basis with  $g(A) = h$  and  $g(A - \{x\}) = G_1(h) = g$ . Define  $A_1 = \{a - x \mid a \in A \text{ and } a \neq x\}$ . Let  $n$  be any large positive integer. Then

$$n + hx = a_1 + a_2 + \cdots + a_h, \quad \text{where } a_i \in A.$$

Then

$$n = (a_1 - x) + (a_2 - x) + \cdots + (a_h - x).$$

After deleting the “0” terms in the above summation, we see that  $n$  is a sum of at most  $h$  elements from  $A_1$ , i.e.,  $A_1$  is an asymptotic basis of order  $h$ . Since  $g(A - \{x\}) = g$ , we see that

$$n + gx = \sum_{i=1}^g a_i, \quad \text{where } a_i \in A - \{x\}.$$

Hence

$$n = \sum_{i=1}^g (a_i - x), \quad \text{where } a_i - x \in A_1.$$

Thus  $A_1$  is an exact asymptotic basis with  $g(A_1) \leq g$ . In fact, easy to see that  $g(A_1) = g$ . Therefore,  $t(h) \geq g = G_1(h)$ .

On the other hand, assume that  $A$  is an asymptotic basis of order  $h$  so that  $g(A) = t(h)$ . Since  $t(h) > h$ ,  $0 \notin A$ . Define  $A' = A \cup \{0\}$ . Then  $A'$  is an exact asymptotic basis of order  $h$ , i.e.,  $g(A') = h$ . Since  $g(A' - \{0\}) = g(A) = t(h)$ , we have that  $g = G_1(h) \geq t(h)$ . Therefore,  $G_1(h) = t(h)$ .

Noting that  $A = \bigcup_{k=0}^{\infty} (2^{2k}, 2^{2k+1}]$  is an asymptotic basis of order 2 and its exact order is 3 as an exact asymptotic basis, we see that  $G_1(2) \geq 3$ . The following are the known exact values for the function:

$G_1(2) = 3$	(Erdős and Graham [5], 1980),
$G_1(3) = 7$	(Nash [46], 1985),
$G_1(4) = 10$	(Li [40], 1989, unconfirmed),
$G_1(5) = 15$	(Li [40], 1989, unconfirmed),
$G_2(3) = 13$	(Nash [46], 1985),
$G_k(2) = 2k + 2$	(Nash [46], 1985).

The following is a list of known estimates for  $G_1(h)$  by various authors:

$$\frac{1}{4}h^2 + o(h^2) \leq G_1(h) \leq \frac{5}{4}h^2 + o(h^2) \quad (\text{Erdős and Graham [5], 1980}),$$

$$\frac{1}{3}h^2 + O(h) \leq G_1(h) \leq h^2 + h \quad (\text{Grekos [11], 1982}),$$

$$G_1(h) \leq \frac{1}{2}h^2 + h \quad (\text{Nash [45], 1993}).$$

#### 4 Postage Stamp problem

Support an envelope has space for only up to  $h$  stamps, and  $A = \{a_1 = 1, a_2, \dots, a_k\}$  is the set of stamp face values. The postage stamp problem consists of computing the smallest postage value  $n(h, A) + 1$  that cannot be stamped by using the given stamps. In other words,  $n(h, A)$  is the largest integer such that every positive integer  $\leq n(h, A)$

can be represented as a linear combination  $\sum_{i=1}^k x_i a_i$  with  $x_i \geq 0$  and  $\sum_{i=1}^k x_i \leq h$ . Define, for any given positive integers  $h$  and  $k$ ,

$$n(h, k) = \max_{|A|=k} n(h, A).$$

For convenience, a set  $A \subseteq \{1, 2, \dots, n\}$  is called an  $h$ -basis for  $n$  if  $\{1, 2, \dots, n\} \subseteq h^0 A$ .  $n(h, A)$  is called the  $h$ -range of  $A$ , and  $n(h, k)$  is called the  $(h, k)$ -range. One central problem is to calculate the  $(h, k)$ -range  $n(h, k)$ .

The postage stamp problem has been around for a long time. However, it seems that the problem appeared only as recreational and entertaining mathematics (e.g., Sprague [62], Problem 18 and Legard [39]) until 1937 when Rohrbach [58] first formalized and analyzed the problem mathematically. Since then, there have been extensive research on the problem (see Guy [13], Hofmeister [16], Hofmeister et al. [17], Klotz [35], etc.). A similar and related problem is the *Frobenius Coin Problem* which asks the largest amount of postage that is impossible to pay by using a given set of stamps (sufficient supply). It turns out that this is an incredibly difficult problem to solve. We even do not know the answer with only three kinds of stamps! Computation of Frobenius problem is NP-complete (see an interesting article of Cipra [3] in the *Science* magazine). See a survey up to 1980 by Alter and Barnett [1], and [31] for recent developments on the postage stamp problem. See Selmer [60, 61] for a comprehensive introduction to the problem.

Stöhr [63] proved in 1955 that

$$n(h, 2) = \left\lfloor \frac{h^2 + 6h + 1}{4} \right\rfloor, \quad h \geq 2. \quad (2)$$

Hofmeister [15] proved in 1968 that, for all integers  $h \geq 23$ ,

$$n(h, 3) = \frac{4}{81}h^3 + \frac{2}{3}h^2 + \alpha h + \beta, \quad (3)$$

where  $\alpha$  and  $\beta$  are determined constants depending only on  $h \pmod{9}$ .

The following theorem (see [30]) provides lower bounds for  $G_k(h)$  by using the lower bounds for  $n(h, k)$ .

**Theorem 6.** *Let  $h \geq 3$  and  $k \geq 1$  be integers. Then*

- (i)  $G_k(h) \geq n(h-1, k+1)$ ;
- (ii)  $G_k(h) \geq 2n(h-1, k) + h$ .

For convenience, let us define, for any given positive integer  $h$ ,

$$\sigma_h = \limsup_{k \rightarrow \infty} \frac{n(h, k)}{k^h}.$$

Mrose [43] proved in 1979 that  $\sigma_2 \geq \frac{2}{7} \approx 0.2857$ . As for the upper bound, it is easy to see that  $\sigma \leq \frac{1}{2}$ . This trivial bound has been improved several times:

$\sigma_2 \leq 0.4992$	(1937, Rohrbach [58])
$\sigma_2 \leq 0.4903$	(1960, Moser [41])
$\sigma_2 \leq 0.4867$	(1960, Riddell [55])
$\sigma_2 \leq 0.4847$	(1969, Moser, Pounder and Riddell [42])
$\sigma_2 \leq 0.4802$	(1969, Klotz [35, 36])
$\sigma_2 \leq 0.4789$	(2006, Güntürk and Nathanson [12])
$\sigma_2 \leq 0.4778$	(2007, Horváth [18])
$\sigma_2 \leq 0.4697$	(2009, Yu [69])

For  $h = 3$ , Mrose [43] proved in 1979 that

$$n(h, 3) \geq \frac{32}{27} \left(\frac{k}{3}\right)^3 + O(k^2).$$

This was improved by Windecker (unpublished) to

$$n(3, k) \geq \frac{4}{3} \left(\frac{k}{3}\right)^3 + O(k^2) \quad \text{as } k \rightarrow \infty. \quad (4)$$

Therefore, it follows from the lower bounds and (ii) in Theorem 6 that

$$G_k(3) \geq \frac{4}{7}k^2 + O(k) \quad \text{as } k \rightarrow \infty,$$

and

$$G_k(4) \geq \frac{8}{81}k^3 + O(k^2) \quad \text{as } k \rightarrow \infty.$$

Using the following recursive inequality for  $n(h, k)$

$$n(h_1 + h_2, k_1 + k_2) \geq n(h_1, k_1)n(h_2, k_2), \quad (5)$$

One can prove that, for any fixed  $h \geq 3$ , as  $k \rightarrow \infty$ ,

$$n(h, k) \geq c_h \left(\frac{4}{3}\right)^{\lfloor k/3 \rfloor} \left(\frac{k}{h}\right)^h,$$

where  $c_h$  are absolute constants depending only on  $h \bmod 3$ . It then follows from Theorem 6 (ii) that

$$G_k(h) \geq 2c_{h-1} \left(\frac{k}{h-1}\right)^{h-1}.$$

## 5 Extremal Bases for Finite Cyclic Groups

Let  $A$  be a set of  $k$  distinct integers.  $A$  is called an  $h$ -basis for  $\mathbb{Z}_m$  if every element in  $\mathbb{Z}_m$  can be written as a sum of at most  $h$  not necessarily distinct elements of  $A$ . In other words,  $A$  is an  $h$ -basis for  $\mathbb{Z}_m$  if and only if  $h^0 A = \mathbb{Z}_m$ . Let  $m(h, A)$  denote the largest positive integer  $m$  so that  $A$  is an  $h$ -basis for  $\mathbb{Z}_m$ . Given positive integers  $h$  and  $k$ , we define

$$m(h, k) = \max_{\substack{A \\ |A|=k}} m(h, A)$$

It is easy to see that  $nm(h, k) \geq (h, k) + 1$  for all  $h \geq 1$  and  $k \geq 1$ . Similar to the Theorem of Erdős and Graham, one can prove that a set  $A = \{a_1, a_2, \dots, a_k\}$  is an  $h$ -basis for  $\mathbb{Z}_m$  for some integer  $h$  if and only if

$$\gcd\{m, a_1, a_2, \dots, a_k\} = 1.$$

It is clear that an  $h$ -basis for  $n$  is always an  $h$ -basis for  $\mathbb{Z}_{n+1}$ . However, an  $h$ -basis for  $\mathbb{Z}_m$  may not be an  $h$ -basis for  $m-1$ .

Extremal bases for finite cyclic groups are closely related to interconnection network designs. Extremal bases and related problems have been one of central focuses in the study of combinatorial networks, which emerges as a broad area of research. For more information, see, for instance, Graham and Sloane [8], Erdős and Hsu [6], Du and Hsu [4], Hsu and Jia [19, 20], and Jia [28, 29], etc.

Hsu and Jia [20] proved in 1994 that

$$m(h, 2) = \left\lfloor \frac{h(h+4)}{3} \right\rfloor + 1 \quad \text{for all } h \geq 2. \quad (6)$$

For  $k = 3$ , it seems harder to handle bases for  $\mathbb{Z}_n$  when compared with bases for  $[1, n]$ . The analog of Hofmeister's formula (3) for  $n(h, 3)$  in the postage stamp problem, an exact formula for  $m(h, 3)$ , is yet to be found. Hsu and Jia [20] showed in 1994 that

$$m(h, 3) \geq \frac{1}{16}h^3 + O(h^2) \approx 0.0625h^3 + O(h^2) \quad \text{as } h \rightarrow \infty.$$

It is easy to verify the following recursive addition inequality similar to (5)

$$m(h_1 + h_2, k_1 + k_2) \geq m(h_1, k_1)m(h_2, k_2). \quad (7)$$

Using this addition inequality, we can provide lower bounds for  $m(h, k)$  by improving lower bounds for  $m(h, k)$  with small  $ks$ .

Jia [23] showed in 1990 that, for fixed  $k \geq 4$  as  $h \rightarrow \infty$ ,

$$m(h, k) \geq \alpha_k \left( \frac{256}{125} \right)^{\lfloor k/4 \rfloor} \left( \frac{h}{k} \right)^k + O(h^{k-1}),$$

where  $\alpha_k = 1, 1, \frac{4}{3}$  and  $\frac{27}{16}$  according as  $k \equiv 0, 1, 2$  or  $3 \pmod{4}$ . Chen and Gu [2] proved in 1992 that, for fixed  $k$  and  $h \rightarrow \infty$ ,

$$m(h, k) \geq \beta_k \left( \frac{2048}{625} \right)^{\lfloor k/4 \rfloor} \left( \frac{h}{k} \right)^k + O(h^{k-1}),$$

where  $\beta_k = 1, 1, \frac{4}{3}$ , or  $\frac{135}{64}$ , according as  $k = 0, 1, 2$ , or  $3 \pmod{4}$ . In 1993, Su [65] constructed a new 5-element  $h$ -basis which provides a new lower bound for  $m(h, 5)$ , and hence a better lower bound in the general case:

$$\begin{aligned} m(h, k) &\geq \gamma_k \left( \frac{5^5 \cdot 7^4}{17^5} \right)^{\lfloor k/5 \rfloor} \left( \frac{h}{k} \right)^k + O(h^{k-1}) \\ &\approx \gamma_k (5.2844)^{\lfloor k/5 \rfloor} \left( \frac{h}{k} \right)^k + O(h^{k-1}), \end{aligned} \quad (8)$$

where

$$\gamma_k = \begin{cases} 1 & \text{if } k \equiv 0, 1 \pmod{5} \\ 4/3 & \text{if } k \equiv 2 \pmod{5} \\ \frac{4752}{2197} \approx 2.163 & \text{if } k \equiv 3 \pmod{5} \\ \frac{165888}{50625} = 3.2768 & \text{if } k \equiv 4 \pmod{5} \end{cases}$$

This implies the following lower bound for  $G_k(h)$ , which is best known at the time this article was written:

$$G_k(h) \geq \gamma_{k+1} \left( \frac{5^5 \cdot 7^4}{17^5} \right)^{\lfloor (k+1)/5 \rfloor} \left( \frac{h}{k+1} \right)^{k+1} + O(h^k) \quad \text{as } h \rightarrow \infty$$

## 6 Remarks and Open Problems

1. Kirfel [34] proved in 1990 that the following limit

$$\tau_k = \lim_{h \rightarrow \infty} \frac{n(h, k)}{h^k}$$

exists for every  $k \geq 1$ . It is known that

$$\tau_1 = 1, \quad \tau_2 = \frac{1}{4}, \quad \text{and} \quad \tau_3 = \frac{4}{81}.$$

It seems much harder to deal with  $n(h, k)$  with given  $h \geq 1$  as  $k$  approaches infinity. Current best known bounds for  $n(2, k)$  are proved by Mrose [43] and Yu [69]:

$$0.2857 < \frac{n(2, k)}{k^2} < 0.4697.$$

2. It is natural to ask if any of the following limits exists:

$$\lim_{h \rightarrow \infty} \frac{G_k(h)}{h^{k+1}} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{G_k(h)}{k^{h-1}}.$$

The only known nontrivial case is Nash's formula for  $G_k(2) = 2k + 2$ . We even do not know the answer when  $k = 1$ .

3. Similarly we may ask if the following limits exist:

$$\eta_k = \lim_{h \rightarrow \infty} \frac{m(h, k)}{h^k}.$$

The only known exact values are  $\eta_1 = 1$  and  $\eta_2 = \frac{1}{3}$ . One annoying fact is that we do not even have any nontrivial lower bound<sup>1</sup> or upper bound for  $m(2, k)$ .

4. A set  $A$  of nonnegative integers is called a *restricted exact asymptotic basis* of order  $h$  if every large positive integer can be written as a sum of  $h$  distinct elements from  $A$ . Similar questions can be asked for restricted exact asymptotic bases. We know little about restricted exact asymptotic bases. Not much is known for the restricted version for both  $n(h, k)$  and  $m(h, k)$ , especially  $m(h, k)$ .
5. Let  $A$  be a finite set of integers. If  $A$  is an basis for  $\mathbb{Z}_m$  then the average order of  $A$  for  $\mathbb{Z}_m$  is defined by

$$\lambda(A, \mathbb{Z}_m) = \frac{1}{m} \sum_{t=0}^{m-1} h(t),$$

where  $h(t)$  is the *length* of  $t$  by  $A$ , which is defined as the minimum number of elements (another problem if distinct elements required) of  $A$  with sum  $t$ . Similar functions can be defined:

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<sup>1</sup> The author does not know any lower bound for  $m(2, k)$  other than  $m(2, k) \geq \frac{2}{7}k^2 + O(k)$ , which is obtained by  $n(2, k) + 1 \leq m(2, k)$ .

$$m(\lambda, k) = \max\{m \mid \exists A \text{ with } |A| = k \text{ and } \lambda(A, \mathbb{Z}_m) \leq \lambda\},$$

$$n(\lambda, k) = \max\{n \mid \exists A \text{ with } |A| = k \text{ and } \lambda(A, [1, n]) \leq \lambda\}.$$

See [21, 29, 31] for some preliminary results in these cases.

6. Graham and Sloane [8] studied a set of four extremal functions related to additive bases. Those extremal functions can be generalized [19]. These functions are related to  $G_k(h)$ . There are still many open problems related these functions.
7. Extremal bases for finite cyclic groups are considered as good underlying topology for interconnection networks (see [6, 19]). This area has been studied extensively in recent years, see [31] for a more complete survey.
8. Wong and Coppersmith [68] discovered a geometric representation of bases for finite cyclic groups, which helps establish upper bounds for  $m(k, h)$ . With help of Kneser's theorem, one might be able to utilize Wong-Coppersmith's representation to obtain upper bounds for  $G_k(h)$ . For more information on the geometric representation of bases see Jia and Hsu [31].
9. Bases for finite groups have also been studied extensively in the past. Among many interesting problems in this area, Rohrbach's problem on bases for finite groups attracts a lot of attention. If  $A$  is a basis of order  $h$  for a finite group  $\Gamma$  with  $|\Gamma| = m$ , then

$$|A| \geq m^{1/h} - 1.$$

Rohrbach [58, 59] asked in 1937 the following question: Is it true that, for every positive integer  $h$ , there exists a constant  $c = c(h) > 0$  such that every finite group  $\Gamma$  with  $|\Gamma| = m$  contains a basis  $A$  of order  $h$  for  $\Gamma$  such that

$$|A| \leq cm^{1/h}?$$

This problem is related to short products of elements from a finite group [27]. The question is still largely open. See [23, 26, 31, 38] for recent developments.

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