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A simple construction of minimal asymptotic bases

by

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1. Introduction. Let N be the set of all nonnegative integers. A subset A of N is called an *asymptotic basis of order* h if every sufficiently large integer can be represented as a sum of h not necessarily distinct elements in A. An asymptotic basis A of order h is called *minimal* if no proper subset of A is an asymptotic basis of order h. Stöhr [4] introduced this concept of minimality. Härtter [1] showed by a nonconstructive argument that there exist minimal asymptotic bases. Nathanson [2] constructed the first nontrivial examples of minimal asymptotic bases of order $h \geqslant 2$. In this paper we give a simple and explicit construction of minimal asymptotic bases of order h for every $h \geqslant 2$. In particular, it is proved that if $h \geqslant 2$ and $1/h \leqslant \alpha < 1$, then there exists a minimal asymptotic basis of order h whose counting function has order of magnitude x^{α}.

2. Results. Let W be a subset of N. Denote by $\mathscr{F}^*(W)$ the set of all finite, nonempty subsets of W. Let $A(W)$ be the set of all numbers of the form $\sum_{f \in F} 2^f$, where $F \in \mathscr{F}^*(W)$. Note that $\emptyset \notin \mathscr{F}^*(W)$, hence $0 \notin A(W)$. For any real number x, let $[x]$ denote the greatest integer n such that $n \leqslant x$, and $\langle x \rangle$ the least integer n such that $n \geqslant x$. If A is a subset of N, let hA denote the set of all sums of h elements of A. Let $A(x)$ denote the counting function of A.

THEOREM 1. *Let* $h \geqslant 2$, *and let* $t = \langle \log(h+1)/\log 2 \rangle$. *Partition* N *into* h *pairwise disjoint subsets* $W_0, \ldots, W_{h-1}$ *such that each set* W_r *contains infinitely many intervals of* t *consecutive integers. Then*

$$A = A(W_0) \cup \ldots \cup A(W_{h-1})$$

is a minimal asymptotic basis of order h.

The proof uses the following two lemmas of Nathanson [3].

LEMMA 1. (a) *If* W_1 *and* W_2 *are disjoint subsets of* N, *then* $A(W_1) \cap A(W_2) = \emptyset$.

(b) *If* $W \subseteq N$ *and* $W(x) = \alpha x + O(1)$ *for some* $\alpha \in (0, 1]$, *then there exist positive constants* c_1 *and* c_2 *such that*

$$c_1 x^\alpha < A(W)(x) < c_2 x^\alpha$$

for all x *sufficiently large.*

(c) *Let* $N = W_0 \cup W_1 \cup \ldots \cup W_{h-1}$ *be a partition, where* $W_r \neq \emptyset$ *for* $r = 0, 1, \ldots, h-1$. *Then*

$$A = A(W_0) \cup A(W_1) \cup \ldots \cup A(W_{h-1})$$

is an asymptotic basis of order h. *Indeed,* $hA = \{n \in N: n \geqslant h\}$ *and* $h(A \cup \{0\}) = N$.

LEMMA 2. *Let* $w_1, \ldots, w_s$ *be* s *distinct nonnegative integers. If*

$$\sum_{i=1}^{s} 2^{w_i} = \sum_{j=1}^{t} 2^{x_j},$$

where $x_1, \ldots, x_t$ *are nonnegative integers that are not necessarily distinct, then there is a partition of* $\{1, 2, \ldots, t\}$ *into* s *nonempty sets* $J_1, \ldots, J_s$ *such that*

$$2^{w_i} = \sum_{j \in J_i} 2^{x_j}$$

for $i = 1, 2, \ldots, s$.

Proof of Theorem 1. By Lemma 1, the set A is an asymptotic basis of order h. We must show that A is minimal.

Let $a \in A$. Then $a \in A(W_r)$ for some r. Without loss of generality, we can assume that $a \in A(W_0)$. Then there is a finite, nonempty subset $F \subseteq W_0$ such that

$$a = \sum_{i \in F} 2^i.$$

Let M denote the largest element of F.

Let $a_0 = a$. We shall construct positive integers a_r for $r = 1, 2, \ldots, h-1$. Choose $m(r) \in W_r$ such that $m(r) > M$ and the t consecutive integers $m(r)$, $m(r)+1, \ldots, m(r)+t-1$ belong to W_r. Let F_r be any subset of $(M, m(r)) \cap W_r$. Define a_r by

$$a_r = \sum_{\substack{i \in W_r \\ i < M}} 2^i + \sum_{i \in F_r} 2^i + \sum_{i=m(r)}^{m(r)+t-1} 2^i. \tag{1}$$

Then $a_r \in A(W_r)$ and

$$2^{m(r)} \leqslant a_r < 2^{m(r)+t}.$$

Let $n = a_0 + \ldots + a_{h-1}$. We shall show that this is the unique representation of n as a sum of h elements of A.

Suppose $n = b_0 + \dots + b_{h-1}$, where $b_r \in A$ for $r = 0, \dots, h-1$. Then $b_r \in A(W_{k(r)})$ for some $k(r) \in [0, h-1]$. Suppose there exists $s \in \{1, 2, \dots, h-1\}$ such that $b_r \notin A(W_s)$ for $r = 0, 1, \dots, h-1$. By Lemma 2 there are subsets $U_r \subseteq W_{k(r)}$ such that

$$\sum_{i=m(s)}^{m(s)+t-1} 2^i = \sum_{r=0}^{h-1} \sum_{i \in U_r} 2^i.$$

Clearly, each i in U_r is less than $m(s)$. It follows from the definition of t that

$$2^{m(s)}(2^t - 1) = \sum_{i=m(s)}^{m(s)+t-1} 2^i = \sum_{r=0}^{h-1} \sum_{\substack{i \in U_r \\ i < m(s)}} 2^i$$

$$\leqslant h \sum_{i=0}^{m(s)-1} 2^i < h 2^{m(s)} \leqslant 2^{m(s)}(2^t - 1),$$

which is impossible. Therefore, after suitable renumbering, $b_r \in A(W_r)$ for $r = 1, 2, \dots, h-1$.

Next we show that $b_0 \in A(W_0)$. Suppose $b_0 \notin A(W_0)$. We may assume without loss of generality that $b_0 \in A(W_1)$. Since $b_r \in A(W_r)$, it follows from Lemma 2 that there exist $V_0 \subseteq W_1$ and $V_r \subseteq W_r$ for $r = 1, 2, \dots, h-1$ such that

$$\sum_{r=0}^{h-1} \sum_{i \in V_r} 2^i = a_0 + \sum_{r=1}^{h-1} \sum_{\substack{i \in W_r \\ i < M}} 2^i. \tag{2}$$

Since $i < M$ for all $i \in \bigcup_{r=0}^{h-1} V_r$, it follows that

$$\sum_{r=0}^{h-1} \sum_{i \in V_r} 2^i = \sum_{i \in V_0} 2^i + \sum_{r=1}^{h-1} \sum_{i \in V_r} 2^i$$

$$\leqslant \sum_{i \in V_0} 2^i + \sum_{r=1}^{h-1} \sum_{\substack{i \in W_r \\ i < M}} 2^i < 2^M + \sum_{r=1}^{h-1} \sum_{\substack{i \in W_r \\ i < M}} 2^i \leqslant a_0 + \sum_{r=1}^{h-1} \sum_{\substack{i \in W_r \\ i < M}} 2^i,$$

which contradicts (2). Hence, $b_0 \in A(W_0)$. Since the representation of an integer as a sum of distinct powers of 2 is unique, it follows that $a_r = b_r$ for $r = 0, 1, \dots, h-1$. In particular, $b_0 = a$. This completes the proof.

COROLLARY 1. *Let $N = W_0 \cup W_1$ be a partition such that each W_i contains infinitely many pairs of consecutive integers. Then $A = A(W_0) \cup A(W_1)$ is a minimal asymptotic basis of order* 2.

COROLLARY 2. *Let $N = W_0 \cup W_1 \cup W_2$ be a partition such that each W_i contains infinitely many pairs of consecutive integers. Then $A = A(W_0) \cup A(W_1) \cup A(W_2)$ is a minimal asymptotic basis of order* 3.

These two corollaries are immediate consequences of Theorem 1 with $t = 2$.

LEMMA 3. *Let $t \geqslant 2$ and $h \geqslant 2$. Let $\alpha_0, \ldots, \alpha_{h-1}$ be positive real numbers such that $\alpha_0 + \ldots + \alpha_{h-1} = 1$. Then there exists a partition of N in the form $N = W_0 \cup W_1 \cup \ldots \cup W_{h-1}$ such that, for $r = 0, 1, \ldots, h-1$,*

(i) $W_r(x) = \alpha_r x + O(1)$;
(ii) *W_r contains infinitely many intervals of t consecutive integers;*
(iii) *In W_r, the gaps between successive intervals of length t are bounded.*

Proof. For any integer $n \geqslant 1$, define $a_r(n)$ and R_n by

$$a_r(n) = [n\alpha_r] \quad \text{for } r = 0, 1, \ldots, h-1,$$

$$R_n = \sum_{r=0}^{h-1} a_r(n).$$

Let $\{R_{n(k)}\}_{k=1}^{\infty}$ be a maximal strictly increasing subsequence of $\{R_n\}_{n=1}^{\infty}$. It follows from $\sum_{r=0}^{h-1} \alpha_r = 1$ and the definition of R_n that

(3) $$n(k) < n(k+1) \leqslant n(k) + h,$$

(4) $$R_{n(k)} < R_{n(k+1)} \leqslant R_{n(k)} + h,$$

(5) $$R_{n(k)} \leqslant n(k) < R_{n(k)} + h,$$

$$d_r(k) = a_r(n(k+1)) - a_r(n(k)) = 0 \text{ or } 1.$$

Let $R_{n(k+1)} - R_{n(k)} = u$. Then there are u distinct integers $r_i \in \{0, 1, \ldots, h-1\}$ such that

$$d_{r_1}(k) = \ldots = d_{r_u}(k) = 1.$$

The remaining $h-u$ integers $r_i \in \{0, 1, \ldots, h-1\}$ satisfy

$$d_{r_{u+1}}(k) = \ldots = d_{r_h}(k) = 0.$$

Let $t \geqslant 2$. Define

$$W_{r_i,k} = [(R_{n(k)} + i - 1)t, \ (R_{n(k)} + i)t - 1] \quad \text{for } i = 1, 2, \ldots, u;$$

$$W_{r_i,k} = \emptyset \quad \text{for } i = u+1, \ldots, h.$$

For each $r = 0, 1, \ldots, h-1$, we define

(6) $$W_r = \bigcup_{k=1}^{\infty} W_{r,k}.$$

It is clear that $N = W_0 \cup \ldots \cup W_{h-1}$, that $W_i \cap W_j = \emptyset$ for $i \neq j$, and that each W_r contains infinitely many intervals of length t. It follows from $\alpha_r > 0$ that (iii) holds.

Let $x \geqslant 1$. Suppose that

$$tR_{n(k)} \leqslant x < tR_{n(k+1)}.$$

Then, by (4) and (5), we have

$$|x - tn(k)| < th,$$
$$|x - tn(k+1)| < 2th.$$

Therefore, for each $r = 0, 1, \ldots, h-1$,

$$W_r(x) \leqslant a_r(n(k+1))t = [n(k+1)\alpha_r]t$$
$$\leqslant tn(k+1)\alpha_r < x\alpha_r + 2th\alpha_r,$$
$$W_r(x) \geqslant a_r(n(k))t = [n(k)\alpha_r]t$$
$$> tn(k)\alpha_r - t > x\alpha_r - th\alpha_r - t,$$

and so $W_r(x) = \alpha_r x + O(1)$. This completes the proof.

THEOREM 2. *For every α such that $1/h \leqslant \alpha < 1$, there is a minimal asymptotic basis A of order h such that*

$$c_1 x^\alpha < A(x) < c_2 x^\alpha \tag{7}$$

for all sufficiently large x.

Proof. Let $\alpha_0 = \alpha$, and define $\alpha_r = (1-\alpha)/(h-1)$ for $r = 1, 2, \ldots, h-1$. Then $\alpha_0 + \ldots + \alpha_{h-1} = 1$ and $\alpha_0 \geqslant \alpha_r > 0$ for $r = 1, 2, \ldots, h-1$. Let $t = \langle \log(h+1)/\log 2 \rangle$. By Lemma 3, there is a partition of N in the form $N = W_0 \cup \ldots \cup W_{h-1}$ such that each set W_r contains infinitely many intervals of length t and

$$W_r(x) = \alpha_r x + O(1).$$

Theorem 1 implies that $A = A(W_0) \cup \ldots \cup A(W_{h-1})$ is a minimal asymptotic basis of order h, and Lemma 1 implies that (7) holds for all sufficiently large x. This completes the proof.

THEOREM 3. *Let $h \geqslant 2$ and let $t = \langle \log(h+1)/\log 2 \rangle$. Let $\alpha_0, \ldots, \alpha_{h-1}$ be positive real numbers such that $\alpha_0 + \ldots + \alpha_{h-1} = 1$. Let $N = W_0 \cup \ldots \cup W_{h-1}$ be a partition satisfying conditions* (i), (ii), *and* (iii) *of Lemma* 3. *Let $A = A(W_0) \cup \ldots \cup A(W_{h-1})$, and let $a \in A$. Define $E_a = hA \setminus h(A \setminus \{a\})$. If $a \in A(W_r)$ and $\alpha = \alpha_r$, then*

$$E_a(x) \gg x^{1-\alpha}.$$

Proof. Condition (iii) implies that there is an integer L such that in every interval $(y-L, y-1]$ there are t consecutive integers belonging to W_r for each $r = 0, 1, \ldots, h-1$.

Let $a \in A$. Without loss of generality we can assume that $a \in A(W_0)$. We must show that $E_a(x) \gg x^{1-\alpha_0}$.

Let 2^M be the largest power of 2 that appears in the binary representation of a. Let x be a large positive number, and let $y = (\log x)/\log 2$. The interval $(y-L, y-1]$ contains integers $m(1), m(2), \ldots, m(h-1)$ such that $m(r)+j \in (y-L, y-1] \cap W_r$ for $r = 1, 2, \ldots, h-1$ and $j = 0, 1, \ldots, t-1$. Let $F_r \subseteq (M, y-L] \cap W_r$. Define a_r by (1). Let $n = a + a_1 + \ldots + a_{h-1}$. Then $n < 2^y = x$. The proof of Theorem 1 shows that $n \in hA \setminus h(A \setminus \{a\}) = E_a$, and that different choices of the $h-1$ sets $F_1, \ldots, F_{h-1}$ lead to different numbers n. Since there are $2^{W_r(y-L)-W_r(M)}$ choices of the set F_r, it follows that the number of n determined by $F_1, \ldots, F_{h-1}$ is

$$\prod_{r=1}^{h-1} 2^{W_r(y-L)-W_r(M)} \geqslant 2^{-M} 2^{\sum_{r=1}^{h-1} W_r(y-L)}$$

$$= 2^{-M} 2^{\sum_{r=1}^{h-1} (\alpha_r y + O(1))}$$

$$\gg (2^y)^{\sum_{r=1}^{h-1} \alpha_r} = (2^y)^{1-\alpha_0} = x^{1-\alpha_0}.$$

Therefore, $E_a(x) \gg x^{1-\alpha_0}$. This completes the proof.

An asymptotic basis A of order h is called *strongly minimal* if $E_a(x) \gg (A(x))^{h-1}$ for each $a \in A$ and for all x sufficiently large.

COROLLARY 3. *Let A satisfy the conditions of Theorem 3. If $\alpha_r = 1/h$ for $r = 0, 1, \ldots, h-1$, then A is a strongly minimal asymptotic basis of order h.*

Proof. Since $A(x) \ll x^{1/h}$, the result follows immediately from Theorem 3.

3. Open problems

1. For $h = 2$, find all partitions $N = W_0 \cup W_1$ such that $A = A(W_0) \cup A(W_1)$ is a minimal asymptotic basis of order 2. Nathanson [3] has constructed an example of a partition of N into two disjoint sets that does not produce a minimal asymptotic basis of order 2.

2. If $N = W_0 \cup W_1 \cup \ldots \cup W_{h-1}$ is a partition such that $w \in W_r$ implies either $w-1 \in W_r$ or $w+1 \in W_r$, then is $A = A(W_0) \cup A(W_1) \cup \ldots \cup A(W_{h-1})$ a minimal asymptotic basis of order h?

3. It would be interesting to extend the results of this paper to asymptotic bases constructed from partitions of N by means of g-adic representations for $g \geqslant 3$.

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