

REPRESENTATIVES FOR FINITE SETS

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ABSTRACT. This paper considers a combinatorial problem by M. B. Nathanson [1], concerning simultaneous systems of representatives for two families of finite sets.

1. INTRODUCTION

Let $\mathcal{S} = \{S_i\}$ be a family of nonempty sets. The set X is a system of representatives for \mathcal{S} if $X \cap S_i \neq \emptyset$ for every S_i in \mathcal{S} . If X is a system of representatives for \mathcal{S} but no proper subset of X is a system of representatives for \mathcal{S} , then X is called a minimal system of representatives for \mathcal{S} . By $D(\mathcal{S})$ we denote the number of minimal systems of representatives for \mathcal{S} . Let $|S_i|$ denote the cardinality of the set S_i . If \mathcal{S} consists of s pairwise disjoint sets S_i with $|S_i| = h$ for all i , then $D(\mathcal{S}) = h^s$.

Let $\mathcal{S} = \{S_i\}$ and $\mathcal{T} = \{T_j\}$ be two families of nonempty sets. A set X is called a simultaneous system of representatives for \mathcal{S} and \mathcal{T} if X is a minimal system of representatives for \mathcal{S} and X is also a system of representatives for \mathcal{T} . $N(\mathcal{S}, \mathcal{T})$ denotes the number of the simultaneous systems of representatives for \mathcal{S} and \mathcal{T} . The study of the numbers $D(\mathcal{S})$ and $N(\mathcal{S}, \mathcal{T})$ could be usefully applied to investigate minimal asymptotic bases in additive number theory [2].

In 1985, Nathanson [1] asked the following question:

Let $h \geq 2$ and $k \geq 1$. Does there exist a real number $\mu = \mu(h, k) \in (0, 1)$ such that

$$N(\mathcal{S}, \mathcal{T}) \leq D(\mathcal{S})\mu^t$$

holds for any families \mathcal{S} and \mathcal{T} of sets satisfying the following properties?

- (i) $\mathcal{S} = \{S_i\}$ is a family of s nonempty, distinct sets S_i with $|S_i| \leq h$ for all i ;
- (ii) $\mathcal{T} = \{T_j\}$ is a family of t nonempty, pairwise disjoint sets T_j with $|T_j| \leq k$ for all j ;
- (iii) S_i is not a subset of T_j for all i and j .

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In this paper, it is proved that no such real number μ exists for any $h \geq 2$ and any $k \geq 1$. Adding some further restriction on \mathcal{S} , we prove that such μ exists in a special case.

2. MAIN RESULTS

Theorem 1. *Let $h \geq 2$ and $k \geq 1$. For any real number $\mu \in (0, 1)$, there exist two families of sets*

$$\mathcal{S} = \{S_i; i = 1, \dots, s\} \quad \text{and} \quad \mathcal{T} = \{T_j; j = 1, \dots, t\}$$

satisfying the following properties:

- (i) $0 < |S_i| \leq h$ for all i ;
- (ii) $0 < |T_j| \leq k$ for all j ;
- (iii) $T_j \cap T_{j'} = \emptyset$ for all $j \neq j'$;
- (iv) S_i is not contained in T_j for all i and j ;
- (v) $N(\mathcal{S}, \mathcal{T}) > D(\mathcal{S})\mu^t$.

Proof. Let μ be any real number so that $0 < \mu < 1$. Let t be an integer such that $\mu^t < 1/h$, and let $s = tk$. Let $a_1, \dots, a_{h-1}, b_1, \dots, b_t$ be $h-1+s$ different elements. Define

$$S_i = \{a_1, \dots, a_{h-1}, b_i\}, \quad T_j = \{b_{(j-1)k+1}, \dots, b_{jk}\}$$

for $i = 1, 2, \dots, s$ and $j = 1, 2, \dots, t$. Let

$$\mathcal{S} = \{S_i; i = 1, 2, \dots, s\}, \quad \mathcal{T} = \{T_j; j = 1, 2, \dots, t\}$$

It is clear that

$$N(\mathcal{S}, \mathcal{T}) = 1, \quad D(\mathcal{S}) = h - 1 + 1 = h.$$

Therefore we have

$$N(\mathcal{S}, \mathcal{T}) = 1 > h\mu^t = D(\mathcal{S})\mu^t,$$

which proves the theorem.

Theorem 1 means that the answer to the question is negative for any $h \geq 2$ and $k \geq 1$. However, we have the following result:

Theorem 2. *Let $h \geq 2$. Suppose*

- (i) $\mathcal{S} = \{S_i\}$ is a family of nonempty, distinct sets S_i with $|S_i| \leq h$ for all i ;
- (ii) Every S_i intersects at most one S_j in \mathcal{S} other than S_i itself;
- (iii) $\mathcal{T} = \{T_j\}$ is a family of t sets T_j with $T_j = \{a_j\}$ for all j , where the a_j 's are distinct elements;
- (iv) S_i is not contained in T_j for any i and j .

Then

$$(1) \quad N(\mathcal{S}, \mathcal{F}) \leq D(\mathcal{S})(1 - 1/h)^{t/2}.$$

Proof. By induction on t for any fixed s . If $t = 0$, then

$$N(\mathcal{S}, \mathcal{F}) = D(\mathcal{S}),$$

hence (1) holds for $t = 0$ and any s . Let $t \geq 1$. Assume that (1) holds for any s and any $t' < t$.

Let

$$\mathcal{S} = \{S_i; i = 1, 2, \dots, s\} \quad \text{and} \quad \mathcal{F} = \{T_j; j = 1, 2, \dots, t\}$$

be two families of sets satisfying the conditions (i)-(iv). If there exists some $T_{j'} = \{a_j\}$ such that $a_j \notin S_i$ for all i , then $N(\mathcal{S}, \mathcal{F}) = 0$, hence (1) holds for t and any s . Now we assume that

$$S = \bigcup_{i=1}^s S_i \supseteq \{a_1, \dots, a_t\}.$$

We consider $T_{j'} = \{a_j\}$. Then the following three cases may occur.

Case I. There exists an i' such that $a_j \in S_{i'}$, where $S_{i'} \cap S_i = \emptyset$ for all $i \neq i'$. $S_{i'} \not\subseteq T_{j'}$ implies that $|S_{i'}| \geq 2$. It is readily verified that

$$\mathcal{S}' = \mathcal{S} \setminus \{S_{i'}\} \quad \text{and} \quad \mathcal{F}' = \{T_j; j = 1, \dots, t-1\}$$

satisfy the conditions (i)-(iv), and

$$D(\mathcal{S}') = |S_{i'}| D(\mathcal{S}').$$

If X is a simultaneous system of representatives for \mathcal{S} and \mathcal{F} , then $X' = X \setminus \{a_j\}$ is a simultaneous system of representatives for \mathcal{S}' and \mathcal{F}' . Conversely, if X' is a simultaneous system of representatives for \mathcal{S}' and \mathcal{F}' , then $X = X' \cup \{a_j\}$ is a simultaneous system of representatives for \mathcal{S} and \mathcal{F} . Therefore

$$\begin{aligned} N(\mathcal{S}, \mathcal{F}) &= N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-1)/2} \\ &= (1/|S_{i'}|) D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \\ &\leq \frac{1}{2} D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \\ &< D(\mathcal{S})(1 - 1/h)^{t/2}. \end{aligned}$$

Case II. There exists an i' such that $a_j \in S_{i'} \cap S_{i''}$ for some i'' . It follows from (ii) that

$$(S_{i'} \cap S_{i''}) \cap S_i = \emptyset$$

for any $i \neq i'$ and $i \neq i''$. Let

$$|S_{i'} \cap S_{i''}| = r, \quad |S_{i'} \setminus S_{i''}| = u, \quad |S_{i''} \setminus S_{i'}| = v.$$

Since $a_i \in S_{i'} \cap S_{i''}$, it is clear that if X is a simultaneous system of representatives for \mathcal{S} and \mathcal{F} , then $X \setminus \{a_i\}$ is a simultaneous system of representatives for

$$\mathcal{S}' = \mathcal{S} \setminus \{S_{i'}, S_{i''}\} \quad \text{and} \quad \mathcal{F}' = \{T_j : j = 1, 2, \dots, t-1\}.$$

Conversely, if X' is a simultaneous system of representatives for \mathcal{S}' and \mathcal{F}' , then $X = X' \cup \{a_i\}$ is a simultaneous system of representatives for \mathcal{S} and \mathcal{F} . Hence

$$N(\mathcal{S}, \mathcal{F}) = N(\mathcal{S}', \mathcal{F}').$$

It is clear that $D(\mathcal{S}) = (r + uv)D(\mathcal{S}')$. (iv) implies that $r + u \geq 2$ and $r + v \geq 2$, thus $\frac{1}{r + uv} \leq 1 - 1/h$. Therefore

$$\begin{aligned} N(\mathcal{S}, \mathcal{F}) &= N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-1)/2} \\ &= \frac{1}{r + uv} D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \\ &\leq (1 - 1/h) D(\mathcal{S})(1 - 1/h)^{(t-1)/2} \\ &< D(\mathcal{S})(1 - 1/h)^{t/2} \end{aligned}$$

Case III. There exists an i' such that

$$a_i \in S_{i'} \setminus S_{i''} \quad \text{and} \quad S_{i'} \cap S_{i''} \neq \emptyset$$

for some $i'' \neq i'$. Let

$$|S_{i'} \cap S_{i''}| = r, \quad |S_{i'} \setminus S_{i''}| = u, \quad |S_{i''} \setminus S_{i'}| = v.$$

It is clear that if there are two sets T_j such that $T_j \subseteq S_{i''} \setminus S_{i'}$, then $N(\mathcal{S}, \mathcal{F}) = 0$, hence (1) holds. If there exists exactly one $T_j = \{a_j\}$ such that $a_j \in S_{i''} \setminus S_{i'}$, then any simultaneous system X of representatives for \mathcal{S} and \mathcal{F} contains a_i and a_j . Hence X is a simultaneous system of representatives for \mathcal{S} and \mathcal{F} if and only if $X \setminus \{a_i, a_j\}$ is a simultaneous system of representatives for

$$\mathcal{S}' = \mathcal{S} \setminus \{S_{i'}, S_{i''}\} \quad \text{and} \quad \mathcal{F}' = \mathcal{F} \setminus \{T_{i'}, T_j\}.$$

It is easily seen that

$$D(\mathcal{S}) = (r + uv)D(\mathcal{S}').$$

Noticing that $r + uv \geq 2$, we have

$$\begin{aligned} N(\mathcal{S}, \mathcal{F}) &= N(\mathcal{S}', \mathcal{F}') \leq D(\mathcal{S}')(1 - 1/h)^{(t-2)/2} \\ &= \frac{1}{r + uv} D(\mathcal{S})(1 - 1/h)^{(t-2)/2} \\ &\leq \frac{1}{2} D(\mathcal{S})(1 - 1/h)^{(t-2)/2} \\ &= D(\mathcal{S})(1 - 1/h)^{t/2}. \end{aligned}$$

If there does not exist T_j such that $a_j \in S_{i''} \setminus S_{i'}$, i.e., if $(S_{i''} \setminus S_{i'}) \cap T_j = \emptyset$ for all j , then any simultaneous system X of representatives for \mathcal{S} and \mathcal{F}

contains a_i and an element x of $S_{i'} \setminus S_{i''}$, hence $X \setminus \{a_i, x\}$ is a simultaneous system of representatives for

$$\mathcal{F}' = \{S_{i'}, S_{i''}\} \text{ and } \mathcal{F} = \{T_j : j = 1, 2, \dots, t-1\}.$$

Conversely, if X' is a simultaneous system of representatives for \mathcal{F}' and \mathcal{F} , then $X = X' \cup \{a_i, x\}$ is a simultaneous system of representatives for \mathcal{F} and \mathcal{F} for any x in $S_{i'} \setminus S_{i''}$. It follows the fact that $r \geq 1$, $0 \leq v \leq h-1$ and $u \geq 1$ that

$$\frac{v}{r+uv} \leq 1 - \frac{1}{h}.$$

Therefore

$$\begin{aligned} N(\mathcal{F}, \mathcal{F}) &= |S_{i'} \setminus S_{i''}| N(\mathcal{F}', \mathcal{F}') \\ &\leq v D(\mathcal{F}') (1 - 1/h)^{(t-1)/2} \\ &= \frac{v}{r+uv} D(\mathcal{F}') (1 - 1/h)^{(t-1)/2} \\ &\leq (1 - 1/h) D(\mathcal{F}') (1 - 1/h)^{(t-1)/2} \\ &< D(\mathcal{F}') (1 - 1/h)^{t/2}. \end{aligned}$$

This completes the proof.

REFERENCES

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