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SIMULTANEOUS SYSTEMS OF REPRESENTATIVES FOR FINITE FAMILIES OF FINITE SETS

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ABSTRACT. Let $h \geq 2$ and $k \geq 1$. It is proved that if $\mathscr{S} = \{S_i\}_{i=1}^s$ and $\mathscr{T} = \{T_j\}_{j=1}^t$ are two families of nonempty, pairwise disjoint sets such that $|S_i| \leq h$, $|T_j| \leq k$ and $S_i \not\subseteq T_j$ for all i and j, then the number $N(\mathscr{S},\mathscr{T})$ of the sets X such that X is a minimal system of representatives for $\mathscr{S}$ and X is simultaneously a system of representatives for $\mathscr{T}$ that satisfies $N(\mathscr{S},\mathscr{T}) \leq h^s(1-(h-r)/h^{q+1})^t$, where $k = q(h-1)+r$ with $0 \leq r \leq h-2$. This was conjectured by M. B. Nathanson [3] in 1985.

1. Introduction. Let $\mathscr{S} = \{S_i\}$ be a family of nonempty sets. The set X is a system of representatives for $\mathscr{S}$ if $X \cap S_i \neq \varnothing$ for every S_i in $\mathscr{S}$. If X is a system of representatives for $\mathscr{S}$, but no proper subset of X is a system of representatives for $\mathscr{S}$, then X is called a minimal system of representatives for $\mathscr{S}$.

Let $\mathscr{S} = \{S_i\}$ and $\mathscr{T} = \{T_j\}$ be two families of nonempty sets. Let $N(\mathscr{S},\mathscr{T})$ denote the number of sets X such that X is a minimal system of representatives for $\mathscr{S}$ and X is also a system of representatives for $\mathscr{T}$.

The study of the number $N(\mathscr{S},\mathscr{T})$ could be usefully applied to investigate asymptotic bases in additive number theory. In 1985, Nathanson [3] made two conjectures on this number, which can be stated as follows.

CONJECTURE 1. *Let $h \geq 2$ and $k \geq 1$. There exists a real number $\lambda = \lambda(h,k) \in (0,1)$ with the following property: Let $\mathscr{S} = \{S_i\}_{i=1}^s$ be a family of s nonempty, pairwise disjoint sets S_i with $|S_i| \leq h$ for all i. Let $\mathscr{T} = \{T_j\}_{j=1}^t$ be a family of t nonempty, pairwise disjoint sets T_j with $|T_j| \leq k$ for all j. Suppose $S_i \not\subseteq T_j$ for all i and j. Then*

$$N(\mathscr{S},\mathscr{T}) \leq h^s \lambda^t. \tag{1}$$

CONJECTURE 2. *Let $h \geq 2$ and $k \geq 1$. Let $k = q(h-1)+r$, where $q = [k/(h-1)]$ and $0 \leq r \leq h-2$. Define*

$$\lambda^*(h,k) = 1-(h-r)/h^{q+1}.$$

Then $\lambda^*(h,k)$ is the smallest value of λ for which inequality (1) *is true for all families $\mathscr{S}$ and $\mathscr{T}$ that satisfy the conditions of Conjecture* 1.

It has been proved that these two conjectures are true in many special cases. Early in 1979, Erdös and Nathanson [1] proved that the conjectures hold if $h = k = 2$ when they investigated asymptotic additive bases of order 2 in additive number theory. Jia [2] proved in 1986 that the conjectures are true in the case that

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$h = k \geq 2$. In his 1985 paper [**3**], Nathanson proved the conjectures in some special cases. And he proved that if $\lambda \in (0,1)$ satisfies $N(\mathscr{S},\mathscr{T}) \leq h^s\lambda^t$ for all $\mathscr{S}$ and $\mathscr{T}$ that satisfy the conditions of Conjecture 1, then $\lambda \geq \lambda^*(h,k)$. In the present paper, we prove that Conjectures 1 and 2 are true for any $h \geq 2$ and $k \geq 1$.

2. Main result and a lemma. The main result of this paper is

THEOREM. *Let $h \geq 2$ and $k \geq 1$. Let $k = q(h-1) + r$, where $q = [k/(h-1)]$ and $0 \leq r \leq h-2$. Then*

$$N(\mathscr{S},\mathscr{T}) \leq h^s(1 - (h-r)/h^{q+1})^t \tag{2}$$

holds for any finite families $\mathscr{S}$ and $\mathscr{T}$ that satisfy the conditions of Conjecture 1.

In particular, we have

$$N(\mathscr{S},\mathscr{T}) \leq h^s((h^2 - h + 1)/h^2)^t$$

if $h = k \geq 2$, which is a result by Jia [**2**], and

$$N(\mathscr{S},\mathscr{T}) \leq h^s(k/h)^t$$

if $h > k \geq 1$.

The following lemma will be used in the proof of the theorem.

LEMMA. *Let $h \geq 2$ and $m \geq 1$ with $m \leq L < mh$. If*

$$L - m = u(h-1) - r, \tag{3}$$

where u is an integer and $0 \leq r \leq h-2$, then

$$x_1 \cdots x_m \geq h^{u-1}(h-r) \tag{4}$$

holds for any integers $1 \leq x_i \leq h$ $(i = 1, 2, \ldots, m)$ with $\sum_{i=1}^m x_i = L$.

PROOF. Let $f(x_1,\ldots,x_m) = x_1 \cdots x_m$. It is well known that f has no minimal point inside the inner $\mathscr{D}$ of the domain $\overline{\mathscr{D}}$: $1 \leq x_i \leq h$ $(i = 1,\ldots,m)$ with the restriction $x_1 + \cdots + x_m = L$. Hence the minimal point of f must be on the boundary $\partial\overline{\mathscr{D}}$ of $\overline{\mathscr{D}}$.

Since $L < mh$, it follows from the definition of u that $u \leq m$. First we assume $u = m$, then $L = mh - r$. We prove

$$f(x_1,\ldots,x_m) \geq h^{m-1}(h-r) \tag{5}$$

by induction on m. It is clear that (5) is true if $m = 1$. Now assume that (5) holds for any $m' < m$. Let $(x_1,\ldots,x_m)$ be a minimal point of f on $\partial\overline{\mathscr{D}}$, where $x_1 \leq x_2 \leq \cdots \leq x_m$. Since $x_1 = L - (x_2 + \cdots + x_m) \geq L - (m-1)h = h - r \geq 2$, it follows from $(x_1,\ldots,x_m) \in \partial\overline{\mathscr{D}}$ that $x_m = h$. Therefore $x_1 + \cdots + x_{m-1} = L - r = (m-1)h - r$, thus

$$\begin{aligned} f(x_1,\ldots,x_m) &= x_1 \cdots x_m = hx_1 \cdots x_{m-1} \\ &\geq h(h^{m-2}(h-r)) = h^{m-1}(h-r), \end{aligned}$$

which proves (5).

Now assume $u < m$. If $(x_1, \ldots, x_m) \in \partial\overline{\mathscr{D}}$ is such that $x_1 \leq \cdots \leq x_m$ and $f(x_1, \ldots, x_m)$ is minimal, then $x_1 = 1$. Otherwise, we suppose $2 \leq x_1 \leq \cdots \leq x_v < x_{v+1} = \cdots = x_m = h$. Then $(x_1 - 1, x_2, \ldots, x_{v-1}, x_v + 1, x_{v+1}, \ldots, x_m) \in \partial\overline{\mathscr{D}}$, and

$$\begin{aligned} f(x_1 - 1, x_2, \ldots, x_{v-1}, x_v + 1, x_{v+1}, \ldots, x_m) &= (x_1 - 1)x_2 \cdots x_{v-1}(x_v + 1)h^{m-v} \\ &= x_1 x_2 \cdots x_v h^{m-v} - x_2 \cdots x_{v-1} h^{m-v}(x_v + 1 - x_1) < f(x_1, x_2, \ldots, x_m), \end{aligned}$$

which contradicts the minimality of $f(x_1, \ldots, x_m)$. Therefore $x_2 + \cdots + x_m = L - 1 = u(h-1) - r + (m-1)$, thus $x_1 \cdots x_m = x_2 \cdots h^{u-1}(h-r)$. This shows that we can assume that $u = m$. Hence the proof of the lemma is complete.

3. The proof of the theorem. Let $\mathscr{S} = \{S_i\}_{i=1}^{s}$ and $\mathscr{T} = \{T_j\}_{j=1}^{t}$ be two finite families of finite sets that satisfy the conditions of Conjecture 1. Let S_{s+1} be a set of h elements such that S_{s+1} does not intersect any S_i in $\mathscr{S}$. Taking $\mathscr{S}' = \mathscr{S} \cup \{S_{s+1}\}$, we have $N(\mathscr{S}', \mathscr{T}) \geq hN(\mathscr{S}, \mathscr{T})$. This allows us to assume that the integer s is sufficiently large. Therefore we may assume without loss of generality that

$$\begin{aligned} |S_i| &= h \quad \text{for } i = 1, 2, \ldots, s; \\ |T_j| &= k \quad \text{for } j = 1, 2, \ldots, t; \end{aligned}$$

and

$$S = \bigcup_{i=1}^{s} S_i \supseteq T = \bigcup_{j=1}^{t} T_j.$$

We will prove the theorem by induction on t for fixed $s > 2kt$. If $t = 0$ then $N(\mathscr{S}, \mathscr{T}) = h^s$. Let $t \geq 1$ and assume that (2) holds for any $0 \leq t' < t$ and any s.

We consider T_t. Let $\{S_1, \ldots, S_m\}$ be the set of those S_i that intersect T_t. Denote

$$|S_i \cap T_t| = n_i \quad \text{for } i = 1, \ldots, m,$$

then $n_1 + \cdots + n_m = k$, and $S_i \not\subseteq T_t$ implies that $1 \leq n_i \leq h - 1$ for $i = 1, \ldots, m$. Let $S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\}$, where $a_{i1}, \ldots, a_{in_i} \in T_t$ for $i = 1, \ldots, m$. Since $s > 2kt$, there exist m S_i in $\mathscr{S}$, say $S_{m+1}, \ldots, S_{2m}$, such that $S_i \cap T = \emptyset$ for $i = m+1, \ldots, 2m$. Let $S_i = \{a_{i1}, a_{i2}, \ldots, a_{ih}\}$ for $i = m+1, \ldots, 2m$.

We construct

$$\begin{aligned} S_i' &= \{a_{i1}, \ldots, a_{in_i}, a_{m+i,n_i+1}, \ldots, a_{m+i,h}\}, \\ S_{m+i}' &= \{a_{m+i,1}, \ldots, a_{m+i,n_i}, a_{i,n_i+1}, \ldots, a_{ih}\} \end{aligned}$$

for $i = 1, \ldots, m$. Let

$$\mathscr{S}' = (\mathscr{S} \backslash \{S_1, \ldots, S_{2m}\}) \cup \{S_1', \ldots, S_{2m}'\}.$$

Then $\mathscr{S}'$ and $\mathscr{T}$ satisfy the conditions of Conjecture 1, and the corresponding integers s and t do not change.

Let X be a simultaneous system of representatives counted in $N(\mathscr{S}, \mathscr{T})$. Denote

$$X \cap S_i = \{x_i\} \quad \text{for } i = 1, \ldots, m;$$

$$Y \backslash \bigcup_{i=1}^{2m} S_i.$$

Then it follows from $S_i \cap T = \varnothing$ for $i = m+1, \ldots, 2m$ that exactly h^m simultaneous systems X of representatives counted in $N(\mathscr{S}, \mathscr{T})$ contain $\{x_1, \ldots, x_m\} \cup X_1$.

Suppose $x_j \in S'_{i_j}$ for $j = 1, \ldots, m$. Then $i_j = j$ or $j + m$, and $S'_{i_j+m} \cap (\{x_1, \ldots, x_m\} \cup X_1) = \varnothing$ for $j = 1, \ldots, m$, where the subscripts of S'_i's are regarded as elements of the group $\mathbf{Z}/(2m)$. Therefore for any $x_{j+m} \in S'_{i_j+m}$ for $j = 1, \ldots, m$, taking $X' = \{x_1, \ldots, x_{2m}\} \cup X_1$, we see that X' is a minimal system of representatives for $\mathscr{S}'$ that contains a system of representatives for $\mathscr{T}$, and X' contains $\{x_1, \ldots, x_m\} \cup X_1$. Since there are h^m different simultaneous systems X of representatives counted in $N(\mathscr{S}', \mathscr{T})$. Therefore $N(\mathscr{S}', \mathscr{T}) \leq N(\mathscr{S}, \mathscr{T})$. Hence we may assume that $S_i \cap T_t = S_i \cap T$ for $i = 1, 2, \ldots, m$.

Let $\mathscr{S}' = \{S_i\}_{i=m+1}^{s}$ and $\mathscr{T}' = \{T_j\}_{j=1}^{t-1}$. Clearly $\mathscr{S}'$ and $\mathscr{T}'$ satisfy the conditions of Conjecture 1 (for $s-m$ and $t-1$). For any X_1 counted in $N(\mathscr{S}', \mathscr{T}')$, there are $h^m - (h - n_1) \cdots (h - n_m)$ different X counted in $N(\mathscr{S}, \mathscr{T})$ containing X_1. Since

$$L = \sum_{i=1}^{m} (h - n_1) = mh - \sum_{i=1}^{m} n_i = mh - k,$$

then $L - m = m(h-1) - k = m(h-1) - q(h-1) - r = (m-q)(h-1) - r$. Hence by the lemma, we have $(h - n_1) \cdots (h - n_m) \geq h^{m-q-1}(h - r)$. Therefore

$$\begin{aligned} N(\mathscr{S}, \mathscr{T}) &\leq (h^m - (h - n_1) \cdots (h - n_m)) N(\mathscr{S}', \mathscr{T}') \\ &\leq (h^m - h^{m-q-1}(h - r)) h^{s-m} (1 - (h - r)/h^{q+1})^{t-1} \\ &= h^s (1 - (h - r)/h^{q+1})^t. \end{aligned}$$

This completes the proof of the theorem.

Nathanson [3] has given an example of two families $\mathscr{S}$ and $\mathscr{T}$ of finite sets that satisfy the conditions of Conjecture 1, for which,

$$N(\mathscr{S}, \mathscr{T}) = h^s (1 - (h - r)/h^{q+1})^t.$$

Therefore the upper bound of $N(\mathscr{S}, \mathscr{T})$ in the theorem above is the best possible result.

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